ASYMPTOTIC SOLUTIONS OF NONLINEAR SECOND ORDER DIFFERENTIAL EQUATIONS WITH VARIABLE COEFFICIENTS

(ASIMPTOTICHESKIE RESHENIIA NELINEINYKH DIFFERENTSIAL'NYKH URAVNENII UTOROGO PORIADKA S PEREMENNYMI KOEFFITSIENTAMI)

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In this paper there are considered the asymptotic solutions of the equation

$$\frac{d^2y}{dt^2} + \varepsilon f(\tau, y)\frac{dy}{dt} + F(\tau, y) = 0 \tag{0.1}$$

where ϵ is a small parameter, $\tau = \epsilon t$ is slow time; by asymptotic solutions are meant the principal terms of asymptotic expansions of solutions.

Investigations of the asymptotic behavior of solutions of differential equations were made in the publications [1-5] and others. The studies [6-9] were devoted directly to the investigation of the asymptotic behavior of the solution of equation (0.1) in certain special cases.

1. Method of computing asymptotic solutions. We shall look for a solution of equation (0.1) in the form

$$y = y(\tau, \omega) \tag{1.1}$$

bearing in mind the equation which connects τ and ω with t, we may write

$$\frac{d\tau}{dt} = \varepsilon, \qquad \frac{d\omega}{dt} = \varphi(\tau) \tag{1.2}$$

In order that the variable ω may increase with an increase in t, we require that the function $\phi(\tau)$ shall not take on negative values. We call attention to the fact that if t changes over the interval $0 \leqslant t \leqslant \tau_0/\epsilon$, then τ and ω vary, in accordance with (1.2), over the interval $0 \leqslant \tau \leqslant \tau_0$, and $0 \leqslant \omega \leqslant \tau_0 \max \phi(\tau)/\epsilon < \infty$. Let us evaluate the derivatives of $y(\tau, \omega)$ with respect to t. Making use of (1.1) and (1.2), we obtain

$$\frac{dy}{dt} = \frac{\partial y}{\partial \omega} \varphi(\tau) + \frac{\partial y}{\partial \tau} \varepsilon$$
$$\frac{d^2 y}{dt^2} = \frac{\partial^2 y}{\partial \omega^2} \varphi^2(\tau) + 2 \frac{\partial^2 y}{\partial \omega\, \partial \tau} \varphi(\tau) \varepsilon + \frac{\partial y}{\partial \omega} \varphi'(\tau) \varepsilon + \frac{\partial^2 y}{\partial \tau^2} \varepsilon^2 \tag{1.3}$$

The prime indicates differentiation with respect to τ. In view of (1.3), equation (0.1) can be put into the form

$$\varphi^2(\tau) \frac{\partial^2 y}{\partial \omega^2} + F(\tau, y) + \varepsilon \Big[2\varphi(\tau) \frac{\partial^2 y}{\partial \omega\, \partial \tau} + \varphi'(\tau) \frac{\partial y}{\partial \omega} +$$
$$+ f(\tau, y)\, \varphi(\tau) \frac{\partial y}{\partial \omega} \Big] + \varepsilon^2 \Big[\frac{\partial^2 y}{\partial \tau^2} + f(\tau, y) \frac{\partial y}{\partial \tau} \Big] = 0 \tag{1.4}$$

Thus, the ordinary differential equation (0.1) has been changed into a partial differential equation. Such a procedure has been used earlier in the works [2] and [5] for the study of quasilinear equations. This method was applied to a nonlinear equation, which represented a special case of equation (0.1), in paper [9].

In a number of cases, and in particular for the one considered here, such an approach makes it possible to shorten the derivations considerably.

In order that equation (1.4) may be satisfied with a degree of precision up to the terms of order $O(\epsilon^2)$, we represent $y(\tau, \omega)$ in the form

$$y(\tau, \omega) = y_0(\tau, \omega) + \varepsilon y_1(\tau, \omega) \tag{1.5}$$

One may write

$$F(\tau, y_0 + \varepsilon y_1) = F(\tau, y_0) + F_y(\tau, y_0) y_1 \varepsilon + \tfrac{1}{2} F_{yy}(\tau, y_0 + \xi y_1) y_1^2 \varepsilon^2 \quad (0 \leqslant \zeta \leqslant \varepsilon)$$
$$f(\tau, y_0 + \varepsilon y_1) = f(\tau, y_0) + f_y(\tau, y_0 + \eta y_1) y_1 \varepsilon \qquad (0 \leqslant \eta \leqslant \varepsilon) \tag{1.6}$$

Substituting the expressions (1.5) and (1.6) into (1.4) we obtain

$$\varphi_2(\tau) \frac{\partial^2 y_0}{\partial \omega^2} + F(\tau, y_0) + \varepsilon \Big\{ \varphi^2(\tau) \frac{\partial^2 y_1}{\partial \omega^2} + F_y(\tau, y_0) y_1 + 2\varphi(\tau) \frac{\partial^2 y_0}{\partial \tau\, \partial \omega} +$$
$$+ [\varphi'(\tau) + f(\tau, y_0) \varphi(\tau)] \frac{\partial y_0}{\partial \omega} \Big\} + \varepsilon \Delta \Big(\tau, \frac{\partial^{k+l} y_0}{\partial \tau^k \partial \omega^l}, \frac{\partial^{k+l} y_1}{\partial \tau^k \partial \omega^l}, \varepsilon \Big) = 0$$
$$(k, l = 0, 1, 2) \tag{1.7}$$

where

$$\Delta \Big(\tau, \frac{\partial^{k+l} y_0}{\partial \tau^k \partial \omega^l}, \frac{\partial^{k+l} y_1}{\partial \tau^k \partial \omega^l}, \varepsilon \Big) = \frac{\partial^2 y_0}{\partial \tau^2} + f(\tau, y_0) \frac{\partial y_0}{\partial \tau} + 2\varphi(\tau) \frac{\partial^2 y_1}{\partial \omega\, \partial \tau} +$$

$$+ [\varphi'(\tau) + f(\tau, y_0)\varphi(\tau)]\frac{\partial y_1}{\partial\omega} + f_y(\tau, y_0 + \eta y_1)\, y_1\varphi(\tau)\frac{\partial y_0}{\partial\omega} +$$
$$+ \frac{1}{2}F_{yy}(\tau, y_0 + \xi y_1)\, y_1^2 + \varepsilon\left[\frac{\partial^2 y_1}{\partial\tau^2} + f(\tau, y_0)\frac{\partial y_1}{\partial\tau} + f_y(\tau, y_0 + \eta y_1)\, y_1\frac{\partial y_0}{\partial\tau} +\right.$$
$$\left.+ f_y(\tau, y_0 + \eta y_1)\, y_1\varphi(\tau)\frac{\partial y_1}{\partial\omega}\right] + \varepsilon^2 f_y(\tau, y_0 + \eta y_1)\, y_1\frac{\partial y_1}{\partial\tau} \qquad (1.8)$$

Putting the coefficients of ϵ^0 and of ϵ equal to zero, we have

$$\varphi^2(\tau)\frac{\partial^2 y_0}{\partial\omega^2} + F(\tau, y_0) = 0 \qquad (1.9)$$

$$\varphi^2(\tau)\frac{\partial^2 y_1}{\partial\omega^2} + F_y(\tau, y_0)\, y_1 = -2\varphi(\tau)\frac{\partial^2 y_0}{\partial\omega\,\partial\tau} - [\varphi'(\tau) + f(\tau, y_0)\varphi(\tau)]\frac{\partial y_0}{\partial\omega} \qquad (1.10)$$

Let us suppose that the functions $y_0(\tau, \omega)$ and $y_1(\tau, \omega)$ can be determined, by means of equations (1.9) and (1.10), as periodic functions of ω with period T_ω independent of τ. We shall prove that under this condition the function (1.8) will be bounded by a constant, independent of ϵ, if $0 < \tau < \tau_0$ and $0 < \omega < \infty$.

For the proof we assume that the functions $F(\tau, y)$ and $f(\tau, y)$ possess a sufficient number of derivatives with respect to τ and y if $0 < \tau < \tau_0$, and $0 < |y| < h$. From known theorems on the existence of differentiable solutions of differential equations it follows that under the given condition the functions $y_0(\tau, \omega)$ and $y_1(\tau, \omega)$, and also their derivatives with respect to τ and ω, up to at least the second order, will be bounded by constants independent of ϵ, if $0 < \tau < \tau_0$, $0 < \omega < T_\omega$, provided the function $\phi(\tau)$ does not vanish and has a sufficient number of derivatives. From the fact that the period of the functions $y_0(\tau, \omega)$ and $y_1(\tau, \omega)$ is independent of τ it follows that their derivatives with respect to τ and ω are also periodic in ω and of the same period. Therefore, it follows from the boundedness of the functions

$$\frac{\partial^{k+l} y_0}{\partial\tau^k\,\partial\omega^l}, \qquad \frac{\partial^{k+l} y_1}{\partial\tau^k\,\partial\omega^l} \qquad (k, l = 0, 1, 2) \qquad (1.11)$$

when $0 < \tau < \tau_0$, and $0 < \omega < T_\omega$, that these functions are bounded also for the values $0 < \tau < \tau_0$ and $0 < \omega < \infty$, and hence also for $0 < t < \tau_0/\epsilon$.

Since the coefficient of ϵ^2 in (1.7) can be expressed by means of the functions (1.11), it follows that this coefficient is bounded if $0 < t < \tau_0/\epsilon$. The following theorem has therefore been proved.

Theorem. If for $0 < \tau < \tau_0$ and $0 < |y| < h$ the functions $f(\tau, y)$ and $F(\tau, y)$ are sufficiently smooth, if the function $\phi(\tau)$ for $0 < \tau < \tau_0$ does not vanish and has a sufficient number of derivatives, and if the functions $y_0(\tau, \omega)$ and $y_1(\tau, \omega)$ can be determined from equations (1.9) and (1.10) as periodic functions of a period which is independent of τ, then the function

$$y(t) = y_0\left(\varepsilon t, \int \varphi(\varepsilon t)\, dt\right) + \varepsilon y_1\left(\varepsilon t, \int \varphi(\varepsilon t)\, dt\right) \tag{1.12}$$

will satisfy equation (0.1) with a precision up to terms of the order of ϵ^2, when $0 < t < \tau_0/\epsilon$.

Let us now pass to the solution of the equations (1.9) and (1.10). In accordance with the established theorem, the functions $y_0(\tau, \omega)$ and $y_1(\tau, \omega)$ have to be determined as functions of ω with a period independent of τ. In regard to the equation (1.9) we shall simply assume that it has a solution $y_0(\tau, \omega)$ which is periodic in ω and of period T_ω independent of τ. Thus, we limit ourselves to the cases in which the equation (0.1) has a first approximation a family of periodic curves. Let us denote by a_1 and a_2 successive zeroes of the derivative $\partial y_0/\partial \omega$. We write down the equations:

$$y_0(\tau, \omega)\big|_{\omega=a_i-\rho} = y_0(\tau, \omega)\big|_{\omega=a_i+\rho}$$

$$\left(\frac{\partial y_0}{\partial \omega}\right)^2\bigg|_{\omega=a_i-\rho} = \left(\frac{\partial y_0}{\partial \omega}\right)^2\bigg|_{\omega=a_i+\rho} \qquad (0 \leqslant \rho \leqslant {}^1/_2\, T_\omega) \quad (i = 1, 2) \tag{1.13}$$

$$\frac{\partial^2 y_0}{\partial \omega\, \partial \tau}\frac{\partial y_0}{\partial \omega}\bigg|_{\omega=a_i-\rho} = -\frac{\partial^2 y_0}{\partial \omega\, \partial \tau}\frac{\partial y_0}{\partial \omega}\bigg|_{\omega=a_i+\rho}$$

For the purpose of proving the validity of these equations, it is necessary to establish the correctness of only the first one of them, since the remaining equations can be obtained from the first one by simple differentiation. Multiplying equation (1.9) by $\partial y_0/\partial \omega$ and integrating the result with respect to ω from a_i to ω, we obtain

$$\frac{\varphi^2(\tau)}{2}\left(\frac{\partial y_0}{\partial \omega}\right)^2 + \int\limits_{y_0(a_i)}^{y_0} F(\tau, y_0)\, dy_0 = 0$$

Hence

$$\omega - a_i = \pm \rho = \pm \left| \int\limits_{y_0(a_i)}^{y_0} \frac{dy_0}{\sqrt{\Phi(\tau, y_0)}} \right|, \qquad \Phi(\tau, y_0) = \left(\frac{\partial y_0}{\partial \omega}\right)^2$$

If ω lies between a_1 and a_2, then it is obvious that the integral in the last equation represents a monotone function of y_0, when y_0 lies between $y_0(a_1)$ and $y_0(a_2)$. Furthermore, if y_0 varies over this interval, then ρ, obviously, will lie between 0 and $a_2 - a_1 = T_\omega/2$. Thus, to each value of y_0 in the mentioned interval there will correspond two values of ω equal to $a_i + \rho$ and $a_i - \rho$. This statement is obviously equivalent to the first one of the equations (1.13), whose truth has thus been established.

We shall look for an expression of $y_1(\tau, \omega)$ in the form

$$y_1(\tau, \omega) = \frac{\partial y_0}{\partial \omega} \int\limits_{a_1 - \frac{1}{2}T_\omega}^{\omega} E(\tau, \omega)\, d\omega \tag{1.14}$$

Here $E(\tau, \omega)$ is a new unknown function. Substituting (1.14) into (1.10), we obtain

$$E(\tau, \omega) = -\frac{1}{\varphi^2(\tau)(\partial y_0/\partial\omega)^2} \int\limits_{a_1}^{\omega} \Big\{ 2\varphi(\tau) \frac{\partial^2 y_0}{\partial\omega\,\partial\tau} \frac{\partial y_0}{\partial\omega} + \\ + [\varphi'(\tau) + f(\tau, y_0)\varphi(\tau)] \Big(\frac{\partial y_0}{\partial\omega}\Big)^2 \Big\} d\omega \tag{1.15}$$

In order that $y_1(\tau, \omega)$ may satisfy the conditions of the established theorem, $y_1(\tau, \omega)$ has to be a periodic function in ω of period T_ω. We therefore require that $E(\tau, \omega) = E(\tau, \omega + T_\omega)$. Because of the periodicity of the integrand in (1.15) and because of its symmetry with respect to the point $\omega = a_2$ [see (1.13)], the last requirement will be fulfilled if

$$\int\limits_{a_1}^{a_2} \Big[2\varphi(\tau) \frac{\partial^2 y_0}{\partial\omega\,\partial\tau} \frac{\partial y_0}{\partial\omega} + \varphi'(\tau) \Big(\frac{\partial y_0}{\partial\omega}\Big)^2 + f(\tau, y_0)\varphi(\tau) \Big(\frac{\partial y_0}{\partial\omega}\Big)^2 \Big] d\omega = 0 \tag{1.16}$$

Let us now prove that the function $E(\tau, \omega)$ is bounded. To do this, it is obviously sufficient to prove that this function is bounded for $\omega = a_1$ and $\omega = a_2$ when the denominator in (1.15) becomes zero.

Suppose that for $\omega = a_1$ and $\omega = a_2$ the function $\partial y_0/\partial\omega$ has zeroes of order r_1 and r_2, respectively. Then the numerator in the formula (1.15) will possess zeroes at these points of order $2r + 1$ and $2r_2 + 1$ [see (1.16)], while the denominator will have zeroes of order $2r_1$ and $2r_2$. Therefore, the function $E(\tau, \omega)$ is not only bounded, but is even equal to zero at $\omega = a_1$ and $\omega = a_2$.

From the boundedness and periodicity of the function $E(\tau, \omega)$ and from (1.14) it follows that for the proof of the periodicity of $y_1(\tau, \omega)$ with respect to ω, it is only necessary to establish the validity of the following equation

$$\int\limits_{a_1 - \frac{1}{2}T_\omega}^{a_1 + \frac{1}{2}T_\omega} E(\tau, \omega)\, d\omega = 0 \tag{1.17}$$

From formulas (1.13) and (1.15) we have

$$E(\tau, a_1 - \rho) = -E(\tau, a_1 + \rho) \qquad (0 \leqslant \rho \leqslant \tfrac{1}{2} T_\omega)$$

Obviously, this establishes (1.17). Thus we have proved that if relation (1.16) holds, then the function $y_1(\tau, \omega)$ is periodic in ω and has a period T_ω independent of τ.

We shall call the relation (1.16) the condition of periodicity. This relation can be rewritten as

$$\frac{d}{d\tau}\left[\varphi(\tau)\int_{a_1}^{a_2}\left(\frac{\partial y_0}{\partial\omega}\right)^2 d\omega\right]+\varphi(\tau)\int_{a_1}^{a_2} f(\tau, y_0)\left(\frac{\partial y_0}{\partial\omega}\right)^2 d\omega=0 \tag{1.18}$$

Let us note one particular case when $f(\tau, y) = f(\tau)$. In this case, equation (1.18) can be integrated and written in the form

$$\varphi(\tau)\int_{a_1}^{a_2}\left(\frac{\partial y_0}{\partial\omega}\right)^2 d\omega = D\exp\left(-\int_{\tau_0}^{\tau} f(\tau)\,d\tau\right) \tag{1.19}$$

where D is an arbitrary constant.

Since the terms of order ϵ in (1.12) and (1.3) represent small oscillating corrections to the principal terms, one can usually neglect these terms involving ϵ. Thus, the asymptotic formulas for the solution of equation (0.1) and its derivative [see (1.15)] have the form:

$$y_0(t)=y_0(\tau,\omega),\quad \left(\frac{dy}{dt}\right)_0=\varphi(\tau)\,\frac{\partial y_0}{\partial\omega}\qquad \left(\tau=\varepsilon t,\quad \omega=\omega_0+\int_{t_0}^{t}\varphi(\varepsilon t)\,dt\right) \tag{1.20}$$

where ω_0 is an arbitrary constant. Therefore, for the determination of the asymptotic solution it is necessary to solve a system consisting of the equation (1.9) (which we shall call the "standard" equation) and of the condition of periodicity.

We call attention to the fact that from (1.2) and the equation $y_0(\tau, \omega) = y_0(\tau, \omega + T_\omega)$ it follows that the instantaneous period of oscillation $T(\tau)$, which for the considered class of problems is defined as the distance between two successive maxima or minima, is connected with the function $\phi(t)$ by the formula:

$$T(\tau)=T_\omega/\varphi(\tau) \tag{1.21}$$

It is for this reason that the function $\phi(\tau)$ will be called the instantaneous frequency of oscillations. The above comparisons of the asymptotic solutions with the exact solutions (as well as the established theorem) have shown that the basic condition for the closeness of the asymptotic solution to the exact one is that the difference between the instantaneous frequency $\phi(\tau)$ and zero be small.

2. Computation of the asymptotic solutions when the solution of the "standard" equation is expressible as an elliptic fu. Jacobi function. Let us consider the equation

$$\frac{d^2y}{dt^2} + a(\tau)\,y + b(\tau)\,y^3 = 0 \qquad (2.1)$$

Its "standard" equation has the form

$$\varphi^2(\tau)\,\frac{\partial^2 y_0}{\partial\omega^2} + a(\tau)\,y_0 + b(\tau)\,y_0{}^3 = 0 \qquad (2.2)$$

The solution of this equation can be expressed by means of any one of the elliptic functions of Jacobi: $\mathrm{sn}[K(\nu)\ \omega,\ \nu]$, $\mathrm{cn}[K(\nu)\ \omega,\ \nu]$, $\mathrm{dn}[K(\nu)\ \omega,\ \nu]$. Here $K(\nu)$ is the complete elliptic integral of the first kind.*

Let us express it by means of the elliptic sine function (see [9]):

$$y_{0,1} = A_1(\tau)\,\mathrm{sn}\,[K(\nu_1(\tau))\,\omega_1,\ \nu_1(\tau)] \qquad (2.3)$$

Substituting (2.3) into (2.2) we obtain

$$A_1(\tau)\,\psi_1{}^2(\tau)\,\frac{\partial^2\,\mathrm{sn}\,u}{\partial u^2} + a(\tau)\,A_1(\tau)\,\mathrm{sn}\,u + b(\tau)\,A_1{}^3(\tau)\,\mathrm{sn}^3\,u = 0 \qquad (2.4)$$

Here $\psi_1(\tau) = K(\nu_1(\tau))\,\phi_1(\tau)$, $u = K(\nu_1(\tau))\ \omega_1$. Eliminating $\partial^2\ \mathrm{sn}\ u/\partial u^2$ by means of the equation for the elliptic sine,

$$\frac{\partial^2\,\mathrm{sn}\,u}{\partial u^2} + (1+\nu)\,\mathrm{sn}\,u - 2\nu\,\mathrm{sn}^3\,u = 0$$

we find that

$$[-\psi_1{}^2(\tau)(1+\nu_1(\tau)) + a(\tau)]\,\mathrm{sn}\,u + [\psi_1{}^2(\tau)\,2\nu_1(\tau) + b(\tau)\,A_1{}^2(\tau)]\,\mathrm{sn}^3\,u = 0$$

By equating to zero the coefficients of sn u and $\mathrm{sn}^3 u$, we obtain two relations for the three functions $A_1(\tau)$, $\nu_1(\tau)$ and $\phi_1(\tau)$:

$$\psi_1{}^2(\tau)(1+\nu_1(\tau)) = a(\tau), \qquad \psi_1{}^2(\tau)\,2\nu_1(\tau) = -b(\tau)\,A_1{}^2(\tau) \qquad (2.5)$$

The one still missing equation, we obtain from the condition of periodicity (1.19).

From (2.3) and [10] we have

$$\frac{\partial y_{0,1}}{\partial\omega} = A_1(\tau)\,K(\nu_1(\tau))\,\frac{\partial\,\mathrm{sn}\,u}{\partial u}, \qquad \frac{\partial\,\mathrm{sn}\,u}{\partial u} = \mathrm{cn}\,u\,\mathrm{dn}\,u \qquad (2.6)$$

If we put (2.6) into (1.13) we obtain

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* We denote by ν the square of the modulus. Furthermore, in place of the usual notation for elliptic functions and for the integrals $\mathrm{sn}(u,\ \sqrt{\nu})$, ..., $E(\sqrt{\nu})$ we shall write $\mathrm{sn}(u,\ \nu)$, ..., $E(\nu)$; when we speak of the modulus we shall mean its square.

$$\phi_1(\tau)A_1^2(\tau)L(\nu_1(\tau)) = G \qquad \left(G = \frac{D}{2}\right) \tag{2.7}$$

Here

$$L(\nu) = \int_0^{K(\nu)} \operatorname{cn}^2 u \operatorname{dn}^2 u \, du = \int_0^1 \sqrt{(1-\zeta^2)(1-\nu\zeta^2)}\, d\zeta =$$
$$= \frac{(1+\nu)E(\nu)-(1-\nu)K(\nu)}{3\nu} \qquad (\zeta = \operatorname{sn} u) \tag{2.8}$$

After some simple transformations, equations (2.5) and (2.7) can be written in the form

$$\frac{4\nu_1^2(\tau)L^2(\nu_1(\tau))}{(1+\nu_1(\tau))^3} = p(\tau), \qquad p(\tau) = \frac{G^2b^2(\tau)}{a^3(\tau)}$$

$$\varphi_1(\tau) = \frac{1}{K(\nu_1(\tau))}\sqrt{\frac{a(\tau)}{(1+\nu_1(\tau))}}, \qquad A_1(\tau) = \sqrt{-\frac{a(\tau)\,2\nu_1(\tau)}{b(\tau)[1+\nu_1(\tau)]}} \tag{2.9}$$

With the aid of these relations one can successively compute the solution functions $\nu_1(\tau)$, $\phi_1(\tau)$ and $A_1(\tau)$.

The graph for the solution of the equation for $\nu_1(\tau)$ is represented in Fig. 1.

Let us consider the cases which correspond to the various combinations of the signs of the coefficients $a(\tau)$ and $b(\tau)$.

1. $a(\tau) > 0$, $b(\tau) < 0$. In this case the curve which determines $\nu_1(\tau)$ lies in the quadrant I, since it is seen from (2.5) that $\nu_1(\tau) > 0$. The solution for $\nu_1(\tau)$ exists if $0 < p(\tau) < 2/9$. At the instant $t = t_1$, when $p(\tau) = 2/9$, the asymptotic solution loses the oscillatory character. If $p(\tau) > 2/9$, as well as in the case when $a(\tau) < 0$, $b(\tau) < 0$, equation (2.2) will have no periodic solutions.

2. $a(\tau) > 0$, $b(\tau) > 0$. The curve which determines $\nu_1(\tau)$ in this case will lie in quadrant IV, for $\nu_1(\tau) < 0$ because of (2.5). The solution for $\nu_1(\tau)$ exists if $0 < p(\tau) < \infty$.

3. $a(\tau) < 0$, $b(\tau) > 0$. In this case the curve in Fig. 1 lies in quadrant III, since $\nu_1(\tau) < 0$. The solution for $\nu_1(\tau)$ will exist in this case when $-\infty < p(\tau) < -4/9$.

The values of elliptic functions and elliptic integrals are usually given in tables for $0 < \nu < 1$. Therefore, in cases 2 and 3, when $\nu_1(\tau) < 0$, in order to be able to use the asymptotic solution (2.3), one has to compute first the values of $\operatorname{sn}[K(\nu)\omega, \nu]$ and $K(\nu)$ for $\nu < 0$. One can, hawever, avoid making these computations if one looks for the solution of the "standard" equation (2.2) in the form

$$y_{0,2} = A_2(\tau)\,\mathrm{cn}\,[K(\nu_2(\tau))\,\omega_2,\ \nu_2(\tau)] \tag{2.10}$$

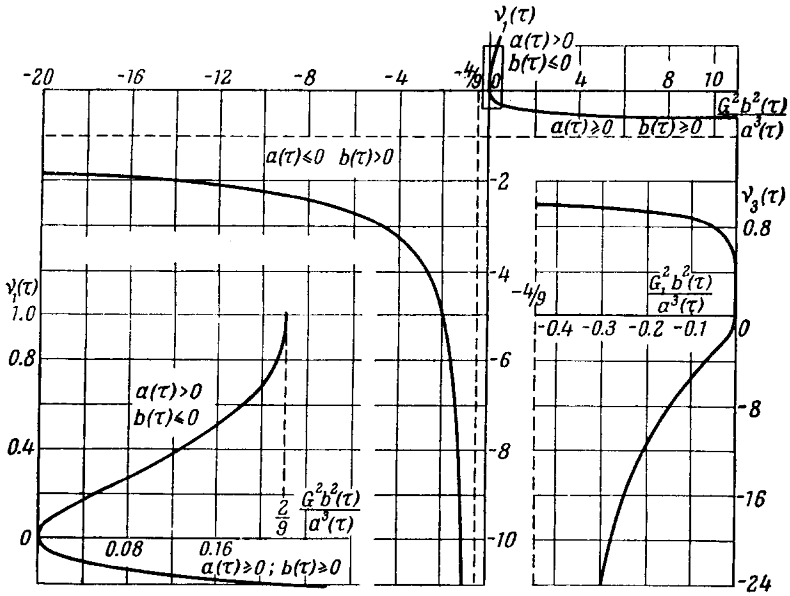

Fig. 1.

Omitting the simple reductions which are entirely analogous to those given above, we give at once the final formulas for the evaluation of the functions $\nu_2(\tau)$, $\phi_2(\tau)$, and $A_2(\tau)$:

$$\frac{4\nu_2^2(\tau)\,M^2(\nu_2(\tau))}{(1-2\nu_2(\tau))^3} = p(\tau) \qquad (\nu_2(\tau)\,b(\tau) \geqslant 0) \tag{2.11}$$

$$\varphi_2(\tau) = \frac{1}{K(\nu_2(\tau))}\sqrt{\frac{a(\tau)}{1-2\nu_2(\tau)}}, \qquad A_2(\tau) = \sqrt{\frac{a(\tau)\,2\nu_2(\tau)}{b(\tau)\,[1-2\nu_2(\tau)]}}$$

Here

$$M(\nu) = \int_0^{K(\nu)} \mathrm{sn}^2 u\,\mathrm{dn}^2 u\,du = \sqrt{(1-\nu)}\,L\left(\frac{-\nu}{1-\nu}\right)$$

The graph for the solution of the equation for the function $\nu_2(t)$ is represented in Fig. 2. Let us consider the cases which can arise for the various combinations of the signs of the functions $a(\tau)$ and $b(\tau)$.

1. $a(\tau) > 0$, $b(\tau) < 0$. The curve in Fig. 2 which corresponds to this case, is located in quadrant IV, and $\nu_2(\tau) < 0$.

2. $a(\tau) > 0$, $b(\tau) > 0$. The curve in Fig. 2, corresponding to this case, is located in quadrant I, and $\nu_2(\tau) > 0$.

3. $a(\tau) < 0$, $b(\tau) > 0$. The corresponding curve in Fig. 2 is located in quadrant II, so that $\nu_2(\tau) > 0$ in this case.

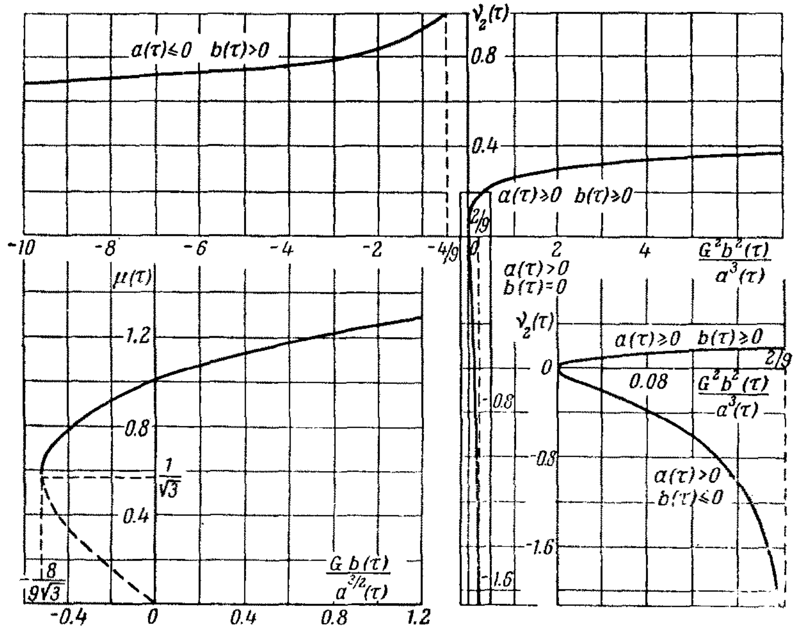

Fig. 2.

Comparing the obtained results, we see that $0 < \nu_2(\tau) < 1$, in cases 2 and 3 if we use the elliptic cosines. Hence we can use the tables in this case. (If we had used the elliptic sine in these cases, then $-\infty < \nu_1(\tau) < 0$).

It is, therefore, more convenient in the cases 2 and 3 to use the asymptotic solutions of (2.10). But in reality the asymptotic solution (2.10) gives nothing new over the asymptotic solution (2.3). Furthermore, one can prove that the asymptotic solutions (2.3) and (2.10) coincide.

There are no real solution for the modulus when $-4/9 < p(\tau) < 0$ as can be seen from Figs. 1 and 2. This is due to the fact that for values of the function $p(\tau)$ which lie in this interval, the solution of (2.2) cannot be expressed in terms of functions which oscillate about the value $y = 0$, as do the elliptic cosine and sine. In this case the solution of equation (2.2) oscillates around the value $y = \pm\sqrt{-a(\tau)/b(\tau)}$.

In order to obtain an asymptotic solution which might exist when the values of $p(\tau)$ are contained in this interval, one must find a solution of the "standard" equation of the form

$$y_{0,3} = A_3(\tau)\,\mathrm{dn}\,[K(\nu_3(\tau))\,\omega_3, \nu_3(\tau)] \tag{2.12}$$

For the functions $\nu_3(\tau)$, $\phi_3(\tau)$, and $A_3(\tau)$ one obtains the following relations

$$\frac{4N^2(\nu_3(\tau))}{(2-\nu_3(\tau))^3} = -p_1(\tau) \qquad \left(p_1(\tau) = \frac{G_1^2 b^2(\tau)}{a^3(\tau)}\right) \tag{2.13}$$

$$\varphi_3(\tau) = \frac{1}{K(\nu_3(\tau))}\sqrt{\frac{-a(\tau)}{2-\nu_3(\tau)}}, \qquad A_3(\tau) = \pm\sqrt{\frac{-2a(\tau)}{b(\tau)[2-\nu_3(\tau)]}}$$

Here

$$N(\nu) = \nu^2 \int_0^{K(\nu)} \mathrm{cn}^2 u \,\mathrm{sn}^2 u \,du = -2\nu^2 \frac{dL}{d\nu}$$

We shall find expressions for the function $A_{3.0}(\tau)$, around which the oscillations take place, and for the amplitude of the oscillations $A_{3.1}(\tau)$. It is known that the function dn u oscillates between 1 and $\sqrt{1-\nu}$. Therefore, by (2.12) we have

$$A_{3.0}(\tau) = A_3(\tau)\,\frac{1+\sqrt{1-\nu_3(\tau)}}{2}, \qquad A_{3.1}(\tau) = \left|A_3(\tau)\,\frac{1-\sqrt{1-\nu_3(\tau)}}{2}\right| \tag{2.14}$$

The graph of the solution of the equation for $\nu_3(\tau)$ is constructed in Fig. 1. It shows that for each value of the function $p_1(\tau)$, which lies in the interval $-4/9 < p_1(\tau) < 0$, there exist two values for $\nu_3(\tau)$, one positive the other negative. One can prove that the asymptotic solutions, which correspond to these values, coincide.

However, due to the fact that the tables of elliptic functions are constructed for positive values of the modulus, one must use the asymptotic solutions which correspond to $0 < \nu_3(\tau) < 1$.

With the aid of the asymptotic solution (2.12), one can extend the solutions (2.3) and (2.10) over an interval of values of t for which $-4/9 < p(\tau) < 0$, and, conversely, by means of the asymptotic solution (2.3) or (2.10) one can extend the solution (2.12) over an interval of values t for which $-\infty < p_1(\tau) < -4/9$, $0 < p_1(\tau) < \infty$. Let us consider briefly the more complicated equation

$$\frac{d^2y}{dt^2} + a_0(\tau) + a_1(\tau)y + a_2(\tau)y^2 + a_3(\tau)y^3 = 0 \tag{2.15}$$

In this case the solution of the "standard" equation should be sought in the form

$$y_0 = \frac{\alpha(\tau)\sigma(\tau,\omega)+\beta(\tau)}{\gamma(\tau)\sigma(\tau,\omega)+1} \quad \text{or} \quad y_0 = \frac{\alpha(\tau)\sigma^2(\tau,\omega)+\beta(\tau)}{\gamma(\tau)\sigma^2(\tau,\omega)+1}$$

Here $\sigma(\tau, \omega)$ can stand for any one of the functions

$$\mathrm{sn}[K(\nu)\omega,\nu], \qquad \mathrm{cn}[K(\nu)\omega,\nu], \qquad \mathrm{dn}[K(\nu)\omega,\nu]$$

One can obtain especially simple formulas in the case when $a_3(r) \equiv 0$. The solution of the standard equation, in this case, should be sought in the form $y_0 = \alpha(r)\sigma^2(r, \omega) + \beta(r)$.

In closing this Section, we remark that in the case when the equations (2.1) and (2.15) contain a term $\epsilon f(r, y)dy/dt$, the asymptotic solutions can also be expressed by means of elliptic Jacobi functions.

An investigation of these cases can be reduced to the study of a single first order differential equation (in the usual modulus ν), whose solution changes $1/\epsilon$ times more slowly than the solution of the original equation, and which can be handled quite simply.

3. Approximate computation of the asymptotic solutions. In those cases when the solution of the "standard" equation cannot be expressed in terms of special functions, we shall seek the solution in the form

$$y_0(\tau, \omega) = \sum_{n=0}^{N} B_n(\tau)\cos n\omega \tag{3.1}$$

Substituting (3.1) into equation (1.1), we obtain

$$-\varphi^2(\tau) \sum_{n=0}^{N} B_n(\tau) n^2 \cos n\omega + \sum_{n=0}^{\infty} F_n[\tau, B_0(\tau), \ldots, B_N(\tau)] \cos n\omega = 0$$

Here

$$F_n[\tau, B_0(\tau), \ldots, B_N(\tau)] = \frac{2}{\pi}\int_0^{\pi} F\left[\tau, \sum_{n=0}^{N} B_n(\tau)\cos n\omega\right]\cos n\omega \, d\omega \tag{3.2}$$

Equating to zero the coefficients of similar harmonics, and neglecting the harmonics of order higher than N, we obtain

$$-\varphi^2(\tau) B_n(\tau) n^2 + F_n[\tau, B_0(\tau), \ldots, B_N(\tau)] = 0 \qquad (n = 0, 1, 2, \ldots, N) \tag{3.3}$$

This system consists of $N + 1$ equations and contains $N + 2$ unknowns: $\phi(r)$, $B_0(r)$, $\ldots$, $B_N(r)$. The required additional equation is obtained by substituting (3.1) into the condition of periodicity (1.18):

$$\frac{d}{d\tau}\left[\varphi(\tau)\frac{1}{2}\pi\sum_{n=0}^{N} B_n^2(\tau) n^2\right] +$$

$$+ \varphi(\tau)\int_0^{\pi} f\left[\tau, \sum_{n=0}^{N} B_n(\tau)\cos n\omega\right]\left[\sum_{n=0}^{N} B_n(\tau) n \sin n\omega\right]^2 d\omega = 0 \tag{3.4}$$

With the aid of the equations (3.3) and (3.4), the functions $\phi(r)$,

$B_0(\tau)$, ..., $B_N(\tau)$ can be expressed in terms of known functions.

The case of greatest interest, due to its simplicity, arises when $N = 1$. In this case the system of equations (3.3) and (3.4) contains only three functions: $\phi(\tau)$, $B_0(\tau)$ and $B_1(\tau)$.

For the purpose of illustrating the method and its precision in the case when $N = 1$, we compute the approximate asymptotic solution of equation (2.1) for the case when $a(\tau) > 0$, and compare it with the exact asymptotic solution. The system of equations (3.3) and (3.4) has the form

$$a(\tau)B_0(\tau) + b(\tau)B_0(\tau)[B_0^2(\tau) + \tfrac{3}{2}B_1^2(\tau)] = 0$$

$$-\varphi^2(\tau)B_1(\tau) + a(\tau)B_1(\tau) + b(\tau)B_1(\tau)[3B_0^2(\tau) + \tfrac{3}{4}B_1^2(\tau)] = 0 \quad (3.5)$$

$$\varphi(\tau)B_1^2(\tau) = C$$

Here C is an arbitrary constant. In the case under consideration, when $a(\tau) > 0$, the oscillations can occur only around $y = 0$. Hence, $B_0(\tau) \equiv 0$. Because of this, the system (3.5) takes on the form:

$$-\varphi^2(\tau) + a(\tau) + \tfrac{3}{4}b(\tau)B_1^2(\tau) = 0, \qquad \varphi(\tau)B_1^2(\tau) = C \qquad (3.6)$$

In order to reduce this system to a simpler form, we introduce the function $\mu(\tau)$ by means of the equation

$$\varphi(\tau) = \sqrt{a(\tau)}\,\mu(\tau) \qquad (3.7)$$

Eliminating the function $B_1(\tau)$ and making use of (3.7), we obtain

$$\mu^3(\tau) - \mu(\tau) - \tfrac{3}{4}q(\tau) = 0 \qquad q(\tau) = cb(\tau)[a(\tau)]^{-3/2} \qquad (3.8)$$

A physical meaning can be ascribed to a real, positive (see Section 1) solution of this equation. Because of (3.6) and (3.7) such a solution will tend to unity as $b(\tau)$ goes to zero. The graph for the computation of such a solution of equation (3.8) is shown in Fig. 2. This figure shows that a solution $\mu(\tau)$ can exist only if $-8\sqrt{3}/9 < q(\tau) < \infty$. At the instant $t = t_1$, when $q(\tau) = -8\sqrt{3}/9$, the solution of equation (2.1) loses its oscillatory nature (a loss of stability occurs within the system).

After the function $\phi(t)$ has been determined, the amplitude of the oscillation $B_1(\tau)$ is computed with the aid of the second one of the equations (3.6).

We shall compare the approximate asymptotic formulas for the instantaneous period, and for the amplitude of the oscillations, with the exact asymptotic formulas (see Section 2).

Let us first consider the case when $b(\tau) = 0$. From (3.6) to (3.8) and

(1.21) we have

$$T(\tau) = \frac{2\pi}{\sqrt{a(\tau)}}, \qquad B_1(\tau) = \sqrt{\frac{C}{\sqrt{a(\tau)}}} \tag{3.9}$$

An examination of equations (2.5) and (2.7) reveals that in this case the exact asymptotic formulas coincide with (3.9).

Let us examine further the case when $b(\tau) < 0$ and $|b(\tau)|$ is increasing. In this case a loss of stability can occur within the system. This case is more difficult to treat by the approximate method when $N = 1$, because the influence of the higher harmonics (which are neglected when $N = 1$) is here most pronounced. Let us compare the approximate and the exact relations at the moment of loss of stability. From the graph in Fig. 2, and from the system (3.6) we have

$$B_1(\tau) = \sqrt{\frac{8}{9}}\sqrt{-\frac{a(\tau)}{b(\tau)}} \tag{3.10}$$

The expression $B_1(\tau)$, which is obtained with the aid of the exact asymptotic solution (see [9]), differs from (3.10) only in so far as in place of the coefficient $\sqrt{8/9} = 0.943$ there stands the number unity. Let us compare the conditions of stability. Just as in [9], we consider the case when

$$y = \alpha, \qquad dy/dt = 0, \qquad b(t) = 0 \qquad \text{at } t = 0$$

From (3.1) and (3.6) we obtain in this case the following results

$$B_1(0) = \alpha, \qquad \varphi(0) = \sqrt{a(0)}, \qquad C = \alpha^2\sqrt{a(0)}$$

For the sake of simplicity let us assume that

$$\min\frac{[a(\tau)]^{3/2}}{|b(\tau)|} = \frac{[\min a(\tau)]^{3/2}}{\max|b(\tau)|} \tag{3.11}$$

The condition of stability takes the form of the inequality

$$\frac{c\,|b(\tau)|}{a(\tau)^{3/2}} < \frac{8}{9\sqrt{3}}$$

Substituting the value of C from (3.11) into this expression, we may rewrite the inequality in the form

$$|\alpha| < \sqrt{\frac{8}{9\sqrt{3}}}\left(\sqrt{\frac{\min a(\tau)}{a(0)}}\,\frac{\min a(\tau)}{\max|b(\tau)|}\right)^{1/2} \tag{3.12}$$

The condition of stability obtained by means of the exact asymptotic solution [9] differs from (3.12) only by the fact that in place of the coefficient $\sqrt{8/9\sqrt{3}} = 0.716$ there stands the coefficient $\sqrt{4\sqrt{2}/3\pi} = 0.775$. Thus, the approximate asymptotic formula compared to the exact asymptotic formula has an error of 10 per cent.

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